

SYMMETRIC CRYSTALS AND AFFINE HECKE ALGEBRAS OF TYPE B

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ABSTRACT. The Lascoux-Leclerc-Thibon conjecture, reformulated and solved by S. Ariki, asserts that the K-group of the representations of the affine Hecke algebras of type A is isomorphic to the algebra of functions on the maximal unipotent subgroup of the group associated with a Lie algebra \mathfrak{g} where \mathfrak{g} is \mathfrak{gl}_∞ or the affine Lie algebra $A_\ell^{(1)}$, and the irreducible representations correspond to the upper global bases. In this note, we formulate analogous conjectures for certain classes of irreducible representations of affine Hecke algebras of type B.

1. INTRODUCTION

The purpose of this note is to formulate and explain conjectures on certain classes of irreducible representations of affine Hecke algebras of type B analogous to the Lascoux-Leclerc-Thibon conjecture ([3]), reformulated and solved by S. Ariki, on affine Hecke algebras of type A.

Let us begin by recalling the Lascoux-Leclerc-Thibon conjecture, reformulated and solved by S. Ariki ([1]). Let H_n^A be the affine Hecke algebra of type A of degree n . Let K_n^A be the Grothendieck group of the abelian category of finite-dimensional H_n^A -modules, and $K^A = \bigoplus_{n \geq 0} K_n^A$. Then it has a structure of Hopf algebra by the restriction and the induction (cf. §3.3). The set $I = \mathbb{C}^*$ may be regarded as a Dynkin diagram with I as the set of vertices and with edges between $a \in I$ and ap_1^2 (see (3.3)). Here p_1 is the parameter of the affine Hecke algebra usually denoted by q . Let \mathfrak{g}_I be the associated Lie algebra, and \mathfrak{g}_I^- the unipotent Lie subalgebra. Let U_I be the group associated to \mathfrak{g}_I^- . Hence \mathfrak{g}_I is isomorphic to a direct sum of copies of $A_\ell^{(1)}$ if p_1^2 is a primitive ℓ -th root of unity and to a direct sum of copies of \mathfrak{gl}_∞ if p_1 has an infinite order. Then $\mathbb{C} \otimes K^A$ is isomorphic to the algebra $\mathcal{O}(U_I)$ of regular functions on U_I . Let $U_q(\mathfrak{g}_I)$ be the associated quantized enveloping algebra. Then $U_q^-(\mathfrak{g}_I)$ has an upper global basis $\{G^{\text{up}}(b)\}_{b \in B(\infty)}$. By specializing $\bigoplus \mathbb{C}[q, q^{-1}]G^{\text{up}}(b)$ at $q = 1$, we obtain $\mathcal{O}(U_I)$. Then the LLT-conjecture says that the elements associated to irreducible H^A -modules corresponds to the image of the upper global basis.

In this note, we shall formulate analogous conjectures for affine Hecke algebras of type B. In the type B case, we have to replace $U_q^-(\mathfrak{g}_I)$ and its upper global basis with a new object, the symmetric crystals (see § 2). It is roughly stated as follows. Let H_n^B be the affine Hecke algebra of type B of degree n . Let K_n^B be the Grothendieck group of the abelian category of finite-dimensional modules over H_n^B , and $K^B = \bigoplus_{n \geq 0} K_n^B$. Then K^B

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has a structure of a Hopf bimodule over K^A . The group U_I has the anti-involution θ induced by the involution $a \mapsto a^{-1}$ of $I = \mathbb{C}^*$. Let U_I^θ be the θ -fixed point set of U_I . Then $\mathcal{O}(U_I^\theta)$ is a quotient ring of $\mathcal{O}(U_I)$. The action of $\mathcal{O}(U_I) \simeq \mathbb{C} \otimes K^A$ on $\mathbb{C} \otimes K^B$, in fact, descends to the action of $\mathcal{O}(U_I^\theta)$.

We introduce $V_\theta(\lambda)$ (see § 2), a kind of the q -analogue of $\mathcal{O}(U_I^\theta)$. Our conjecture is then:

- (i) $V_\theta(\lambda)$ has a crystal basis and a global basis.
- (ii) K^B is isomorphic to a specialization of $V_\theta(\lambda)$ at $q = 1$ as an $\mathcal{O}(U_I)$ -module, and the irreducible representations correspond to the upper global basis of $V_\theta(\lambda)$ at $q = 1$.

We exclude the representations of H_n^B such that X_i have an eigenvalue ± 1 (see § 3).

2. SYMMETRIC CRYSTALS

In this section, we shall introduce crystals associated with quantum groups with an involution.

2.1. Quantized universal enveloping algebras. We shall recall the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let I be an index set (for simple roots), and Q the free \mathbb{Z} -module with a basis $\{\alpha_i\}_{i \in I}$. Let $(\bullet, \bullet): Q \times Q \rightarrow \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}$ for any i and $(\alpha_i^\vee, \alpha_j) \in \mathbb{Z}_{\leq 0}$ for $i \neq j$ where $\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)$. Let q be an indeterminate and set $K := \mathbb{Q}(q)$. We define its subrings \mathbf{A}_0 , \mathbf{A}_∞ and \mathbf{A} as follows.

$$\begin{aligned} \mathbf{A}_0 &= \{f/g; f(q), g(q) \in \mathbb{Q}[q], g(0) \neq 0\}, \\ \mathbf{A}_\infty &= \{f/g; f(q^{-1}), g(q^{-1}) \in \mathbb{Q}[q^{-1}], g(0) \neq 0\}, \\ \mathbf{A} &= \mathbb{Q}[q, q^{-1}]. \end{aligned} \tag{2.1}$$

Definition 2.1. *The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the K -algebra generated by the elements e_i, f_i and invertible elements t_i ($i \in I$) with the following defining relations.*

- (1) *The t_i 's commute with each other.*
- (2) *$t_j e_i t_j^{-1} = q^{(\alpha_j, \alpha_i)} e_i$ and $t_j f_i t_j^{-1} = q^{-(\alpha_j, \alpha_i)} f_i$ for any $i, j \in I$.*
- (3) *$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$. Here $q_i := q^{(\alpha_i, \alpha_i)/2}$.*
- (4) (Serre relation) *For $i \neq j$,*

$$\sum_{k=0}^b (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = \sum_{k=0}^b (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - (\alpha_i^\vee, \alpha_j)$ and

$$\begin{aligned} e_i^{(k)} &= e_i^k / [k]_i! , & f_i^{(k)} &= f_i^k / [k]_i! , \\ [k]_i &= (q_i^k - q_i^{-k}) / (q_i - q_i^{-1}) , & [k]_i! &= [1]_i \cdots [k]_i . \end{aligned}$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i 's (resp. the e_i 's). Let us recall the crystal theory of $U_q^-(\mathfrak{g})$ ([4]). Let e'_i and e_i^* be the

operators on $U_q^-(\mathfrak{g})$ defined by

$$[e_i, a] = \frac{(e_i^* a)t_i - t_i^{-1} e_i' a}{q_i - q_i^{-1}} \quad (a \in U_q^-(\mathfrak{g})).$$

Then these operators satisfy the following formula similar to derivations:

$$e_i'(ab) = e_i'(a)b + (\text{Ad}(t_i)a)e_i'b, \quad e_i^*(ab) = ae_i^*b + (e_i^*a)(\text{Ad}(t_i)b).$$

Then $U_q^-(\mathfrak{g})$ has a unique symmetric bilinear form (\bullet, \bullet) such that $(1, 1) = 1$ and

$$(e_i'a, b) = (a, f_i b) \quad \text{for any } a, b \in U_q^-(\mathfrak{g}).$$

It is non-degenerate and satisfies $(e_i^*a, b) = (a, bf_i)$. The left multiplication of f_j and e_i' have the commutation relation

$$e_i'f_j = q^{-(\alpha_i, \alpha_j)} f_j e_i' + \delta_{ij},$$

and both the e_i' 's and the f_i 's satisfy the Serre relations. Since e_i' and f_i satisfy the q -boson relation, any element $a \in U_q^-(\mathfrak{g})$ can be written uniquely as

$$a = \sum_{n \geq 0} f_i^{(n)} a_n \quad \text{with } e_i' a_n = 0.$$

We define the modified root operators by

$$(2.2) \quad \tilde{e}_i a = \sum_{n \geq 1} f_i^{(n-1)} a_n \quad \text{and} \quad \tilde{f}_i a = \sum_{n \geq 0} f_i^{(n+1)} a_n.$$

Let $L(\infty)$ be the \mathbf{A}_0 -submodule of $U_q^-(\mathfrak{g})$ generated by the $\tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1$'s ($\ell \geq 0, i_1, \dots, i_\ell \in I$). Let us set $B(\infty) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1 \bmod qL(\infty) ; \ell \geq 0, i_1, \dots, i_\ell \in I \} \subset L(\infty)/qL(\infty)$. Then we have

Theorem 2.2. (i) $\tilde{f}_i L(\infty) \subset L(\infty)$ and $\tilde{e}_i L(\infty) \subset L(\infty)$,
(ii) $B(\infty)$ is a basis of $L(\infty)/qL(\infty)$,
(iii) $\tilde{f}_i B(\infty) \subset B(\infty)$ and $\tilde{e}_i B(\infty) \subset B(\infty) \sqcup \{0\}$.

2.2. Global bases. Let $-$ be the automorphism of K sending q to q^{-1} . Then $\overline{\mathbf{A}_0}$ coincides with \mathbf{A}_∞ . Let V be a vector space over K , L_0 an A -submodule of V , L_∞ an \mathbf{A}_∞ -submodule, and $V_{\mathbf{A}}$ an \mathbf{A} -submodule. Set $E := L_0 \cap L_\infty \cap V_{\mathbf{A}}$.

Definition 2.3 ([4]). *We say that $(L_0, L_\infty, V_{\mathbf{A}})$ is balanced if each of L_0 , L_∞ and $V_{\mathbf{A}}$ generates V as a K -vector space, and if one of the following equivalent conditions is satisfied.*

- (i) $E \rightarrow L_0/qL_0$ is an isomorphism,
- (ii) $E \rightarrow L_\infty/q^{-1}L_\infty$ is an isomorphism,
- (iii) $(L_0 \cap V_{\mathbf{A}}) \oplus (q^{-1}L_\infty \cap V_{\mathbf{A}}) \rightarrow V_{\mathbf{A}}$ is an isomorphism.
- (iv) $\mathbf{A}_0 \otimes_{\mathbb{Q}} E \rightarrow L_0$, $\mathbf{A}_\infty \otimes_{\mathbb{Q}} E \rightarrow L_\infty$, $\mathbf{A} \otimes_{\mathbb{Q}} E \rightarrow V_{\mathbf{A}}$ and $K \otimes_{\mathbb{Q}} E \rightarrow V$ are isomorphisms.

Let $-$ be the ring automorphism of $U_q(\mathfrak{g})$ sending q, t_i, e_i, f_i to $q^{-1}, t_i^{-1}, e_i, f_i$.

Let $U_q(\mathfrak{g})_{\mathbf{A}}$ be the \mathbf{A} -subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}, f_i^{(n)}$ and t_i . Similarly we define $U_q^-(\mathfrak{g})_{\mathbf{A}}$.

Theorem 2.4. $(L(\infty), L(\infty)^-, U_q^-(\mathfrak{g})_{\mathbf{A}})$ is balanced.

Let

$$G: L(\infty)/qL(\infty) \xrightarrow{\sim} E := L(\infty) \cap L(\infty)^- \cap U_q^-(\mathfrak{g})_{\mathbf{A}}$$

be the inverse of $E \xrightarrow{\sim} L(\infty)/qL(\infty)$. Then $\{G(b) ; b \in B(\infty)\}$ forms a basis of $U_q^-(\mathfrak{g})$. We call it a (lower) *global basis*. It is first introduced by G. Lusztig ([5]) under the name of “canonical basis” for the A,D,E cases. The dual basis of the lower crystal basis of $U_q^-(\mathfrak{g})$ is called the *upper global basis* of $U_q^-(\mathfrak{g})$.

2.3. Symmetry. Let θ be an automorphism of I such that $\theta^2 = \text{id}$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice Q by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_q(\mathfrak{g})$.

Let $\mathcal{B}_\theta(\mathfrak{g})$ be the K -algebra generated by E_i , F_i , and invertible elements T_i ($i \in I$) satisfying the following defining relations:

$$(2.3) \quad \left\{ \begin{array}{ll} \text{(i)} & \text{the } T_i \text{'s commute with each other,} \\ \text{(ii)} & T_{\theta(i)} = T_i \text{ for any } i, \\ \text{(iii)} & T_i E_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, \alpha_j)} E_j \text{ and } T_i F_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j \text{ for} \\ & i, j \in I, \\ \text{(iv)} & E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i) \text{ for } i, j \in I, \\ \text{(v)} & \text{the } E_i \text{'s and the } F_i \text{'s satisfy the Serre relations.} \end{array} \right.$$

Hence $\mathcal{B}_\theta(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes K[T_i^{\pm 1}; i \in I] \otimes U_q^+(\mathfrak{g})$. We set $E_i^{(n)} = E_i^n / [n]_i!$ and $F_i^{(n)} = F_i^n / [n]_i!$.

Let $\lambda \in P_+ := \{\lambda \in \text{Hom}(Q, \mathbb{Q}) ; \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\}$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

Proposition 2.5. (i) *There exists a $\mathcal{B}_\theta(\mathfrak{g})$ -module $V_\theta(\lambda)$ generated by a vector ϕ_λ such that*

- (a) $E_i \phi_\lambda = 0$ for any $i \in I$,
- (b) $T_i \phi_\lambda = q^{(\alpha_i, \lambda)} \phi_\lambda$ for any $i \in I$,
- (c) $\{u \in V_\theta(\lambda) ; E_i u = 0 \text{ for any } i \in I\} = K \phi_\lambda$.

Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

- (ii) *there exists a unique symmetric bilinear form (\bullet, \bullet) on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$, and it is non-degenerate.*

The pair $(\mathcal{B}_\theta(\mathfrak{g}), V_\theta(\lambda))$ is an analogue of $(\mathcal{B}, U_q^-(\mathfrak{g}))$. Such a $V_\theta(\lambda)$ is constructed as follows. Let $U_q^-(\mathfrak{g})\phi'_\lambda$ and $U_q^-(\mathfrak{g})\phi''_\lambda$ be a copy of a free $U_q^-(\mathfrak{g})$ -module. We give the structure of a $\mathcal{B}_\theta(\mathfrak{g})$ -module on them as follows: for any $i \in I$ and $a \in U_q^-(\mathfrak{g})$

$$(2.4) \quad \begin{aligned} T_i(a\phi'_\lambda) &= q^{(\alpha_i, \lambda)} (\text{Ad}(t_i t_{\theta(i)}) a) \phi'_\lambda, \\ E_i(a\phi'_\lambda) &= (e'_i a + q^{(\alpha_i, \lambda)} \text{Ad}(t_i)(e_{\theta(i)}^* a)) \phi'_\lambda, \\ F_i(a\phi'_\lambda) &= (f_i a) \phi'_\lambda \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} T_i(a\phi''_\lambda) &= q^{(\alpha_i, \lambda)} (\text{Ad}(t_i t_{\theta(i)}) a) \phi''_\lambda, \\ E_i(a\phi''_\lambda) &= (e'_i a) \phi''_\lambda, \\ F_i(a\phi''_\lambda) &= (f_i a + q^{(\alpha_i, \lambda)} (\text{Ad}(t_i) a) f_{\theta(i)}) \phi''_\lambda. \end{aligned}$$

Then there exists a unique $\mathcal{B}_\theta(\mathfrak{g})$ -linear morphism $\psi: U_q^-(\mathfrak{g})\phi'_\lambda \rightarrow U_q^-(\mathfrak{g})\phi''_\lambda$ sending ϕ'_λ to ϕ''_λ . Its image $\psi(U_q^-(\mathfrak{g})\phi'_\lambda)$ is $V_\theta(\lambda)$.

Hereafter we assume further that

$$(2.6) \quad \text{there is no } i \in I \text{ such that } \theta(i) = i.$$

We conjecture that $V_\theta(\lambda)$ has a crystal basis. This means the following. We define the modified root operators similarly to (2.2):

$$\tilde{E}_i(u) = \sum_{n \geq 1} F_i^{(n-1)} u_n \quad \text{and} \quad \tilde{F}_i(u) = \sum_{n \geq 0} F_i^{(n+1)} u_n$$

when writing $u = \sum_{n \geq 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. Let $L_\theta(\lambda)$ be the \mathbf{A}_0 -submodule of $V_\theta(\lambda)$ generated by $\tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi_\lambda$ ($\ell \geq 0$ and $i_1, \dots, i_\ell \in I$), and let $B_\theta(\lambda)$ be the subset $\{\tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi_\lambda \bmod qL_\theta(\lambda); \ell \geq 0, i_1, \dots, i_\ell \in I\}$ of $L_\theta(\lambda)/qL_\theta(\lambda)$.

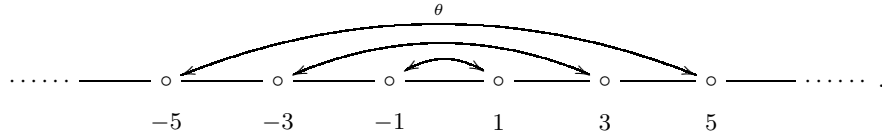
Conjecture 2.6. (i) $\tilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda)$ and $\tilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda)$,
 (ii) $B_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/qL_\theta(\lambda)$,
 (iii) $\tilde{F}_i B_\theta(\lambda) \subset B_\theta(\lambda)$, and $\tilde{E}_i B_\theta(\lambda) \subset B_\theta(\lambda) \sqcup \{0\}$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely, let $-$ be the bar-operator of $V_\theta(\lambda)$ given by $-: a\phi_\lambda \rightarrow \bar{a}\phi_\lambda$ ($a \in U_q^-(\mathfrak{g})$) (such an operator exists).

Conjecture 2.7. $(L_\theta(\lambda), L_\theta(\lambda)^-, U_q^-(\mathfrak{g})_{\mathbf{A}} \phi_\lambda)$ is balanced.

Assume that this conjecture is true. Let $G^{\text{low}}: L_\theta(\lambda)/qL_\theta(\lambda) \xrightarrow{\sim} E := L_\theta(\lambda) \cap L_\theta(\lambda)^- \cap U_q^-(\mathfrak{g})_{\mathbf{A}} \phi_\lambda$ be the inverse of $E \xrightarrow{\sim} L_\theta(\lambda)/qL_\theta(\lambda)$. Then $\{G^{\text{low}}(b); b \in B_\theta(\lambda)\}$ forms a basis of $V_\theta(\lambda)$. We call this basis the *lower global basis* of $V_\theta(\lambda)$. Let $\{G^{\text{up}}(b); b \in B_\theta(\lambda)\}$ be the dual basis to $\{G^{\text{low}}(b); b \in B_\theta(\lambda)\}$ with respect to the inner product of $V_\theta(\lambda)$. We call it the *upper global basis* of $V_\theta(\lambda)$.

We can prove the conjectures in the \mathfrak{gl}_∞ -case:



Theorem 2.8. Let I be the set \mathbb{Z}_{odd} of odd integers. Define

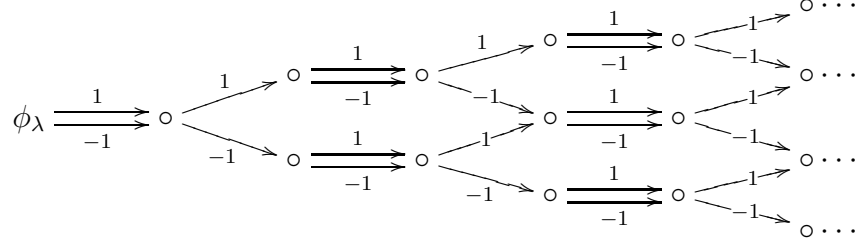
$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 2, \\ 0 & \text{otherwise,} \end{cases}$$

and $\theta(i) = -i$. Then, for $\lambda = 0$, $V_\theta(\lambda)$ has a crystal basis and a global basis.

Note that $\{a \in U_q^-(\mathfrak{g}); a\phi_\lambda = 0\} = \sum_i U_q^-(\mathfrak{g})(f_i - f_{\theta(i)})$ in this case.

The proof is by using a kind of PBW basis, similarly to [5]. The details will appear elsewhere.

The following diagram is the part of the crystal graph of $B_\theta(\lambda)$ that concerns only the 1-arrows and the (-1) -arrows.



Here is the part of the crystal graph of $B_\theta(\lambda)$ that concerns only the n -arrows and the $(-n)$ -arrows for an odd integer $n \geq 3$:

$$\phi_\lambda \xrightleftharpoons[-n]{n} \circ \xrightleftharpoons[-n]{n} \circ \xrightleftharpoons[-n]{n} \circ \xrightleftharpoons[-n]{n} \circ \dots$$

3. AFFINE HECKE ALGEBRA OF TYPE B

3.1. Definition. For $p_0, p_1 \in \mathbb{C}^*$ and $n \in \mathbb{Z}_{\geq 0}$, the affine Hecke algebra H_n^B of type B_n is the \mathbb{C} -algebra generated by T_i ($0 \leq i < n$) and invertible elements X_i ($1 \leq i \leq n$) satisfying the defining relations:

- (i) the X_i 's commute with each other,
- (ii) the T_i 's satisfy the braid relation: $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ ($1 \leq i < n-1$), $T_i T_j = T_j T_i$ ($|i-j| > 1$),
- (iii) $(T_0 - p_0)(T_0 + p_0^{-1}) = 0$ and $(T_i - p_1)(T_i + p_1^{-1}) = 0$ ($1 \leq i < n$).
- (iv) $T_0 X_1^{-1} T_0 = X_1$, $T_i X_i T_i = X_{i+1}$ ($1 \leq i < n$), and $T_i X_j = X_j T_i$ if $j \neq i, i+1$.

We assume that $p_0, p_1 \in \mathbb{C}^*$ satisfy

$$(3.1) \quad p_0^2 \neq 1, p_1^2 \neq 1.$$

Let us denote by $\mathbb{P}ol_n$ the Laurent polynomial ring $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, and by $\widetilde{\mathbb{P}ol}_n$ its quotient field $\mathbb{C}(X_1, \dots, X_n)$. Then H_n^B is isomorphic to the tensor product of $\mathbb{P}ol_n$ and the subalgebra generated by the T_i 's that is isomorphic to the Hecke algebra of type B_n . We have

$$T_i a = (s_i a) T_i + (p_i - p_i^{-1}) \frac{a - s_i a}{1 - X^{-\alpha_i^\vee}} \quad \text{for } a \in \mathbb{P}ol_n.$$

Here $p_i = p_1$ ($1 < i < n$), and $X^{-\alpha_i^\vee} = X_1^{-2}$ ($i = 0$) and $X^{-\alpha_i^\vee} = X_i X_{i+1}^{-1}$ ($1 \leq i < n$). The s_i 's are the Weyl group action on $\mathbb{P}ol_n$: $(s_i a)(X_1, \dots, X_n) = a(X_1^{-1}, X_2, \dots, X_n)$ for $i = 0$ and $(s_i a)(X_1, \dots, X_n) = a(X_1, \dots, X_{i+1}, X_i, \dots, X_n)$ for $1 \leq i < n$.

Note that $H_n^B = \mathbb{C}$ for $n = 0$.

3.2. Intertwiner. The algebra H_n^B acts faithfully on $H_n^B / \sum_i H_n^B (T_i - p_i) \simeq \mathbb{P}ol_n$. Set $\varphi_i = (1 - X^{-\alpha_i^\vee}) T_i - (p_i - p_i^{-1}) \in H_n^B$ and $\tilde{\varphi}_i = (p_i^{-1} - p_i X^{-\alpha_i^\vee})^{-1} \varphi_i \in \widetilde{\mathbb{P}ol}_n \otimes_{\mathbb{P}ol_n} H_n^B$. Then the action of $\tilde{\varphi}_i$ on $\mathbb{P}ol_n$ coincides with s_i . They are called intertwiners.

3.3. Affine Hecke algebra of type A. We will review the LLT conjecture, reformulated and solved by S. Ariki, on the affine Hecke algebras of type A.

The affine Hecke algebra H_n^A of type A_n is isomorphic to the subalgebra of H_n^B generated by T_i ($1 \leq i < n$) and $X_i^{\pm 1}$ ($1 \leq i \leq n$). For a finite-dimensional H_n^A -module M let us decompose

$$(3.2) \quad M = \bigoplus_{a \in (\mathbb{C}^*)^n} M_a$$

where $M_a = \{u \in M; (X_i - a_i)^N u = 0 \text{ for any } i \text{ and } N \gg 0\}$ for $a = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$. For a subset $I \subset \mathbb{C}^*$, we say that M is of type I if all the eigenvalues of X_i belong to I . The group \mathbb{Z} acts on \mathbb{C}^* by $\mathbb{Z} \ni n: a \mapsto ap_1^{2n}$.

Lemma 3.1. *Let I and J be \mathbb{Z} -invariant subsets in \mathbb{C}^* such that $I \cap J = \emptyset$.*

- (i) *If M is an irreducible H_m^A -module of type I and N is an irreducible H_n^A -module of type J , then $\text{Ind}_{H_m^A \otimes H_n^A}^{H_{m+n}^A}(M \otimes N)$ is irreducible of type $I \cup J$.*
- (ii) *Conversely if L is an irreducible H_n^A -module of type $I \cup J$, then there exist m ($0 \leq m \leq n$), an irreducible H_m^A -module M of type I and an irreducible H_{n-m}^A -module N of type J such that L is isomorphic to $\text{Ind}_{H_m^A \otimes H_{n-m}^A}^{H_n^A}(M \otimes N)$.*

Hence in order to study the irreducible modules over the affine Hecke algebras of type A, it is enough to treat the irreducible modules of type I for an orbit I with respect to the \mathbb{Z} -action on \mathbb{C}^* . Let $K_{I,n}^A$ be the Grothendieck group of the abelian category of finite-dimensional H_n^A -modules of type I . We set $K_I^A = \bigoplus_{n \geq 0} K_{I,n}^A$. Then K_I^A has a structure of Hopf algebra where the product and the coproduct

$$\mu: K_{I,m}^A \otimes K_{I,n}^A \rightarrow K_{I,m+n}^A, \quad \Delta: K_{I,n}^A \rightarrow \bigoplus_{i+j=n} K_{I,i}^A \otimes K_{I,j}^A$$

are given by $M \otimes N \mapsto \text{Ind}_{H_m^A \otimes H_n^A}^{H_{m+n}^A}(M \otimes N)$ and by $M \mapsto \text{Res}_{H_i^A \otimes H_j^A}^{H_n^A} M$. Let \mathfrak{g}_I be the Lie algebra associated to the Dynkin diagram with I as the set of vertices and with edges between a and ap_1^2 ($a \in I$). It means

$$(3.3) \quad (\alpha_i, \alpha_j) = 2\delta_{i,j} - \delta_{i,p_1^2 j} - \delta_{p_1^2 i,j} \quad \text{for } i, j \in I.$$

Let U_I be the unipotent group associated with the Lie subalgebra \mathfrak{g}_I^- of \mathfrak{g}_I generated by the f_i 's. Then we have

Lemma 3.2. *Let I be a \mathbb{Z} -invariant set. Then $\mathbb{C} \otimes K_I^A$ is isomorphic to the algebra $\mathcal{O}(U_I)$ of the regular functions on U_I as a Hopf algebra.*

Here, for $a \in I$, f_a corresponds to the one-dimensional H_1^A -module \mathbb{C}_a on which X_1 acts by a . Let $\{G^{\text{up}}(b)\}_{b \in B(\infty)}$ be the upper global basis of $U_q^-(\mathfrak{g})$. Then $(\bigoplus \mathbb{C}[q]G^{\text{up}}(b))/((q-1)\bigoplus \mathbb{C}[q]G^{\text{up}}(b))$ is isomorphic to $\mathcal{O}(U_I)$. The following theorem is conjectured for Hecke algebras of type A by Lascoux-Leclerc-Thibon ([3]) and reformulated and proved by S. Ariki ([1]) for affine Hecke algebras of type A.

Theorem 3.3. *The elements of K^A associated to irreducible H^A -modules correspond to the upper global basis $G^{\text{up}}(b)$ by the isomorphism above.*

Hence the irreducible modules are parametrized by $B(\infty)$. Grojnowski ([2]) constructed the operators \tilde{e}_a and \tilde{f}_a on $B(\infty)$ in terms of irreducible modules. The operator \tilde{e}_a

sends an irreducible H_n^A module M to a unique irreducible submodule of the H_{n-1}^A -module $\{u \in M; (X_n - a)u = 0\}$. The operator \tilde{f}_a sends an irreducible H_n^A module M to a unique irreducible quotient of the H_{n+1}^A -module $\text{Ind}_{H_n^A \otimes H_1^A}^{H_{n+1}^A}(M \otimes \mathbb{C}_a)$.

3.4. Representations of affine Hecke algebras of type B. For $n, m \geq 0$, set $\mathbf{F}_{n,m} := \mathbb{C}[X_1^{\pm 1}, \dots, X_{n+m}^{\pm 1}, D^{-1}]$ where

$$D := \prod_{1 \leq i \leq n < j \leq n+m} (X_i - p_1^2 X_j)(X_i - p_1^{-2} X_j)(X_i - p_1^2 X_j^{-1})(X_i - p_1^{-2} X_j^{-1})(X_i - X_j).$$

Then we can embed H_m^B into $H_{n+m}^B \otimes_{\mathbb{P}ol_{n+m}} \mathbf{F}_{n,m}$ by

$$T_0 \mapsto \tilde{\varphi}_n \cdots \tilde{\varphi}_1 T_0 \tilde{\varphi}_1 \cdots \tilde{\varphi}_n, \quad T_i \mapsto T_{i+n} \quad (1 \leq i < m), \quad X_i \mapsto X_{i+n} \quad (1 \leq i \leq m).$$

Its image commute with $H_n^B \subset H_{n+m}^B$. Hence $H_{n+m}^B \otimes_{\mathbb{P}ol_{n+m}} \mathbf{F}_{n,m}$ is a right $H_n^B \otimes H_m^B$ -module.

Lemma 3.4. $H_{n+m}^A \otimes_{H_n^A \otimes H_m^A} (H_n^B \otimes H_m^B) \otimes_{\mathbb{P}ol_{n+m}} \mathbf{F}_{n,m} \xrightarrow{\sim} H_{n+m}^B \otimes_{\mathbb{P}ol_{n+m}} \mathbf{F}_{n,m}$.

For a finite-dimensional H_n^B -module M , we decompose M as in (3.2). The semidirect product group $\mathbb{Z}_2 \times \mathbb{Z} = \{1, -1\} \times \mathbb{Z}$ acts on \mathbb{C}^* by $(\epsilon, n): a \mapsto a^\epsilon p_1^{2n}$.

Let I and J be $\mathbb{Z}_2 \times \mathbb{Z}$ -invariant subsets of \mathbb{C}^* such that $I \cap J = \emptyset$. Then for an H_n^B -module N of type I and H_m^B -module M of type J , the action of $\mathbb{P}ol_{n+m}$ on $N \otimes M$ extends to an action of $\mathbf{F}_{n,m}$. We set

$$N \diamond M := (H_{n+m}^B \otimes_{\mathbb{P}ol_{n+m}} \mathbf{F}_{n,m}) \otimes_{(H_n^B \otimes H_m^B) \otimes_{\mathbb{P}ol_{n+m}} \mathbf{F}_{n,m}} (N \otimes M).$$

By the lemma above, $N \diamond M$ is isomorphic to $\text{Ind}_{H_n^A \otimes H_m^A}^{H_{n+m}^A}(N \otimes M)$ as an H_{n+m}^A -module.

- Lemma 3.5.** (i) *Let N be an irreducible H_n^B -module of type I and M an irreducible H_m^B -module of type J . Then $N \diamond M$ is an irreducible H_{n+m}^B -module of type $I \cup J$.*
(ii) *Conversely if L is an irreducible H_n^B -module of type $I \cup J$, then there exist an integer m ($0 \leq m \leq n$), an irreducible H_m^B -module N of type I and an irreducible H_{n-m}^B -module M of type J such that $L \simeq N \diamond M$.*
(iii) *Assume that a $\mathbb{Z}_2 \times \mathbb{Z}$ -orbit I decomposes into $I = I_+ \sqcup I_-$ where I_\pm are \mathbb{Z} -orbits and $I_- = (I_+)^{-1}$. Assume that $\pm 1, \pm p \notin I$. Then for any irreducible H_n^B -module L of type I , there exists an irreducible H_n^A -module M such that $L \simeq \text{Ind}_{H_n^A}^{H_n^B} M$.*

Hence in order to study H^B -modules, it is enough to study irreducible modules of type I for a $\mathbb{Z}_2 \times \mathbb{Z}$ -orbit I in \mathbb{C}^* such that I is a \mathbb{Z} -orbit or I contains one of $\pm 1, \pm p$.

In this note, we treat only the case when the $\mathbb{Z}_2 \times \mathbb{Z}$ -orbit I does not contain 1 nor -1 .

For a $\mathbb{Z}_2 \times \mathbb{Z}$ -invariant subset I of \mathbb{C}^* , we define $K_{I,n}^B$ and K_I^B similarly to the case of A -type. Then K_I^B is a (right) Hopf K_I^A -bimodule by the multiplication and the comultiplication

$$\mu: K_{I,n}^B \times K_{I,m}^A \rightarrow K_{I,n+m}^B \quad \text{and} \quad \Delta: K_{I,n}^B \rightarrow \bigoplus_{i+j=n} K_{I,i}^B \otimes K_{I,j}^A$$

given by $L \otimes M \mapsto \text{Ind}_{H_n^B \otimes H_m^A}^{H_{n+m}^B}(L \otimes M)$ and $L \mapsto \text{Res}_{H_i^B \otimes H_j^A}^{H_n^B} L$.

Let θ be the automorphism of I given by $a \mapsto a^{-1}$. Then it induces an automorphism of U_I . Let U_I^θ be the θ -fixed point sets of U_I . Then the action of $\mathcal{O}(U_I) \simeq \mathbb{C} \otimes K_I^A$ on K_I^B descends to an action of $\mathcal{O}(U_I^\theta)$, as it follows from the following lemma.

Lemma 3.6. *For an irreducible H_n^B -module L and an irreducible H_m^A -module M , we have $\mu(L \otimes M) = \mu(L \otimes M^\theta)$, where M^θ is the H_m^A -module induced from M by the automorphism of H_m^A given by $X_i \mapsto X_{m+1-i}^{-1}$, $T_i \mapsto T_{m-i}$.*

Now we take the case

$$I = \{p_1^n; n \in \mathbb{Z}_{\text{odd}}\}.$$

Assume that any of ± 1 and $\pm p_0$ is not contained in I . The set I may be regarded as the set of vertices of a Dynkin diagram by (3.3). Let us define an automorphism θ of I by $a \mapsto a^{-1}$. Let \mathfrak{g}_I be the associated Lie algebra (\mathfrak{g}_I is isomorphic to \mathfrak{gl}_∞ if p_1 has an infinite order, and isomorphic to $A_\ell^{(1)}$ if p_1^2 is a primitive ℓ -th root of unity). Let $V_\theta(\lambda)$ be as in Proposition 2.5 with $\lambda = 0$.

Conjecture 3.7. (i) $V_\theta(\lambda)$ has a crystal basis and a global basis.
 (ii) the elements of K_I^B associated with irreducible representations corresponds to the upper global basis of $V_\theta(\lambda)$ at $q = 1$.

Note that (i) is nothing but Theorem 2.8 when p_1 is not a root of unity.

Let us take the case

$$I = \{p_0 p_1^{2n}; n \in \mathbb{Z}\} \cup \{p_0^{-1} p_1^{2n}; n \in \mathbb{Z}\}.$$

Assume that there exists no integer n such that $p_0^2 = p_1^{4n}$. It includes the case where $p_0 = p_1$ and $p_1^{2n} \neq 1$ for any $n \in \mathbb{Z}_{\text{odd}}$. Let θ be the automorphism of I given by $\theta: a \mapsto a^{-1}$. Then θ has no fixed points. We regard I as the set of vertices of a Dynkin diagram by (3.3). Let \mathfrak{g}_I be the associated Lie algebra. It is isomorphic to either $\mathfrak{gl}_\infty \oplus \mathfrak{gl}_\infty$, \mathfrak{gl}_∞ , $A_\ell^{(1)} \oplus A_\ell^{(1)}$ or $A_\ell^{(1)}$. Set $\lambda = \Lambda_{p_0} + \Lambda_{p_0^{-1}}$ (i.e. $(\alpha_i, \lambda) = \delta_{i, p_0} + \delta_{i, p_0^{-1}}$).

Conjecture 3.8. (i) $V_\theta(\lambda)$ has a crystal basis and a global basis.
 (ii) the elements of K_I^B associated with irreducible representations corresponds to the upper global basis of $V_\theta(\lambda)$ at $q = 1$.

In the both cases, we conjecture that, for an irreducible H_n^B -module M corresponding to an upper global basis $G^{\text{up}}(b)$, $\dim M_a$ coincides with the value of $(\phi_\lambda, E_{a_1} \cdots E_{a_n} G^{\text{up}}(b))$ at $q = 1$ for $a = (a_1, \dots, a_n) \in I^n$.

Miemiets ([6]) introduced the operators \tilde{e}_i and \tilde{f}_i on the set of isomorphic classes of irreducible modules, similarly to the A type case, and studied their properties. We conjecture that they coincide with the operators \tilde{E}_i and \tilde{F}_i on $B_\theta(\lambda)$.

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